A COMPUTATIONAL VALIDATION OF AN ALGORITHM FOR SOLVING A SOCIAL PLANNER'S PROBLEM:

A Case of Linear-Quadratic Approximation to Stochastic Dynamic Programming Problems

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1. INTRODUCTION

The objective of this paper is to describe and validate an algorithm for solving a social planner's problem. The computational details are provided in steps for the computational replication and accuracy of the reported simulation results: (1) numerical results are provided in intermediate steps as well as a final step, and (2) the relative error of the finite computational precision is also provided in a matrix norm. The computational example for this paper is drawn from Díaz-Giménez (1999) which provides the MATLAB software program written by Duran. McCullough and Vinod (1999) reviewed the numerical reliability of econometric software and deplored the implicit reliability of software packages, especially, in the area of non-linear estimation problems. McCullough and Vinod (1999) argue that an analysis of the computational method of running economic models should be verified, not taken for granted. Based on this critique, I reprogrammed the Díaz-Giménez model in C-language with my own sub-routines and verified the reliability of the simulation results. This was done with greater precision (in double precision) than with the results obtained by Duran.

The algorithm focused on in this paper is the linear-quadratic approximation to a nonlinear stochastic dynamic programming problem. This recursive method is also introduced as "the stochastic linear optimal regulator problem" by Sargent (1987). Stokey and Lucas (1989) discuss recursive methods in details. Cooley (ed.) (1995) presents recent advances in dynamic economic theory and computational methods with emphasis on issues in the business cycle. Some advantages in using the linear-quadratic structure have been widely acknowledged: (1) the techniques involved are relatively easy to learn and implement; (2) the programming structure is easily modified for a number of particular applications, such as economies with taxes or any other distortions, or economies with n-period-lived overlapping generations; (3) equilibria can be easily computed even when the dimension of the state variable is large; and (4) an explicit linear policy function can be computed (Hansen and Prescott, 1995). One disadvantage, though, is that the resulting

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equilibrium law of motion must be linear. However, this does not appear to be a serious limitation, since there is little evidence of major nonlinearities in aggregate data (Hansen and Prescott, 1995).

In section 2, the general structure of the model is specified. In section 3, a solution procedure is described. The conclusion follows in section 4. All notations are defined as they appear for the first time in the text.

2. THE GENERAL STRUCTURE OF THE MODEL

2.1 *The major assumptions of the model*

To illustrate an example of the importance of computational aspects, this linear-quadratic approximation method is applied to the well-known neoclassical growth model, which is a basic building block of various business cycle models. Since the main focus is on business cycles, some mechanism to generate cyclical behavior of the economy needs to be built into the original growth model. Cooley and Prescott (1995) stated that the construction and analysis of equilibrium paths for simple artificial economies based on the neoclassical growth model had proven to be a very fruitful approach to studying and better understanding the business cycle.

The major assumptions of the basic neoclassical growth model concern an aggregate household, and an aggregate producer in a country. An aggregate household is assumed, since a large number of identical households live in a country. The life time horizons of the aggregate household is infinite. The aggregate household is assumed to maximize its expected present discounted value of utility (an indication of its well-being) from consumption. An aggregate producer is assumed, since a large number of identical producers are competitively operating in a country. Technology is freely accessible to the producer and can be used to produce output y using capital k available. As soon as output is produced, it can be consumed as either consumption or investment. For this competitive equilibrium model economy, with the application of the second welfare theorem, a benevolent social planner is assumed. The social planner chooses sequences for consumption (a decision variable in this model)$\{c_t\}_{t=0}^{\infty}$, that maximizes his expected present discounted value of utility (an indication of his well-being). In other words, he is deciding how much to consume in period t by equating the cost of not consuming today with the benefit of consuming tomorrow.[1] In this subsequent section, the consumption-production-investment related constraint is substituted into the payoff function. Accordingly, the optimization problem was rewritten with a sequence for investment as a

1) The second welfare theorem implies that the competitive equilibrium allocations are identical to the optimal allocations chosen by a benevolent social planner. The planner's main objective is to maximize the welfare of the representative household.

sequence of decision variable $\{i_t\}_{t=0}^{\infty}$ that maximizes his expected present discounted value of utility.

2.2 *The structure of the model (dynamic programming problem)*

1) Time horizon (T years):

where $T \to \infty$

2) State Variables (z_t, s_t):

2-1) Exogenous state variable (z_t):

where z_t := a technological shock at time t.

2-2) Endogenous state variable (k_t):

where k_t := a capital (stock) at time t.

3) Decision variable (i_t):

where i_t := an investment at time t.

4) Payoff function ($U(c_t)$) :

The utility function can be any well-behaved (time separable) neo-classical utility function, such as a CES function. For simplicity and ease of exposition, a natural log form of a utility function is chosen. The payoff function of this problem is specified as:

$$U(c_t) = \ln c_t \tag{1}$$

where

E := expectation operator,

β := time discount factor, $0<\beta<1$,

U := any well-behaved time separable neoclassical utility function ($\ln c$ is used for simplicity),

c_t := consumption at period t.

5) Constraints:

5-1) Consumption-investment-production relationship:

The production function is assumed to have a well-behaved neoclassical production function. All labor available in a country is completely allocated to productive activities. Labor is not part of the optimal plan of an aggregate household. Therefore, the consumption/investment/production relationship is formally specified as a constraint for the optimization problem as follows:

$$c_t + i_t = F(k_t, z_t) \tag{2}$$
$$= e^{z_t} k_t^{\alpha} \qquad \because F(k_t, z_t) = e^{z_t} k_t^{\alpha}.$$

where

$F(.)$:= a well-behaved neoclassical production function.

z_t := a technological shock at time t,

5-2) Equation of motion:

Capital k_t depreciates at the rate of δ , but it is also accumulated through investment i_t. The equation of motion for capital (stock) is specified as:

$$k_{t+1} = (1-\delta)k_t + i_t \tag{3}$$

where

i_t := investment at time t,

k_t := capital (stock) at time t,

δ := depreciation rate of capital, $0<\delta<1$.

5-3) First order Markov process of a series of technological shock:

Variable z_t indicates a technological shock which is the source of uncertainty in the economy, and it is assumed to follow a first-order linear Markov process (the AR(1) process) specified as:

$$z_{t+1} = \rho z_t + \varepsilon_{t+1} \qquad \because |\rho| < 1. \tag{4}$$

where

L := a linear function of z_t

z_t := a technological shock at time t,

ε_t := a finite sequence of independently and identically distributed random variables with mean $E[\varepsilon_t] = 0$, variance $E[\varepsilon_t^2] = \sigma^2$, and covariance

$E[\varepsilon_t, \varepsilon_s] = 0, \quad \forall t \neq s,$

ρ := a autoregressive parameter. When the condition $|\rho| < 1$ is satisfied, the first-order autoregressive process is stationary.[2)]

2) In other words, the mean, the variance, and the covariances of the ε_t do not change over time.

6) Stochastic dynamic optimization problem:

By substituting one of the constraints $c_t = e_t^{z} k_t^{\alpha} - i_t$ from equation (2) into the payoff function $U(c_t) = \ln c_t$, write the optimization problem as:

$$\underset{i_t}{Max} \ E\sum_{t=0}^{\infty} \beta^t \ln(e_t^{z_t} k_t^{\alpha} - i_t) \tag{5}$$

$$\text{subject to } k_{t+1} = (1-\delta)k_t + i_i,$$
$$z_{t+1} = \rho z_t + \varepsilon_{t+1},$$
$$k_0 \text{ and } z_0 \text{ given.}$$

7) The value function for repeated iteration on the Bellman's equation (a stochastic dynamic programming approach)[3]:

$$V_{n+1}(k_t, z_t) = \underset{i_t}{Max} \left(\ln(e_t^{z_t} k_t^{\alpha} - i_t) + \beta E[v_n(k_{t+1}, z_{t+1}) \mid z_t]\right) \tag{6}$$

$$\text{subject to } k_{t+1} = (1-\delta)k_t + i_i,$$
$$z_{t+1} = \rho z_t + \varepsilon_{t+1}.$$
$$k_0 \text{ and } z_0 \text{ given.}$$

The limiting value function $V = \lim_{n \to \infty} V_n$ has to satisfy the above maximization problem subject to the dynamically specified constraints under various regularity conditions, which are discussed in detail in Stokey and Lucas (1989) and Sargent (1987). The dynamic programming approach is based on Bellman's (1957) principle of optimality.[4] This principle transforms a many period dynamic optimization problem, such as the one stated in equation (5) into many successive one-period optimization problems. This transformation is achieved by simply solving Bellman's equation formulated in equation (6) in a backward recursive manner from the terminal period. In other words, the social planner has no incentive to deviate from the original optimal plan over time. This optimal plan is said to be "time-invariant", which is a routine practice in economics (Sargent, 1987).

3) There are many different approaches to solving dynamic optimization problems. Dynamic programming making use of Bellman's principle of optimality is one approach.

4) Bellman(1957) eloquently described the principle of optimality this way: "An optimal policy has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

(3-5) The Hessian matrix evaluated at the steady state $(\bar{z}, \bar{k}, \bar{i})$ is:

$$\bar{H} \equiv \begin{bmatrix} \frac{\partial^2 \Pi}{\partial z^2} & \frac{\partial^2 \Pi}{\partial z \partial k} & \frac{\partial^2 \Pi}{\partial z \partial i} \\ \frac{\partial^2 \Pi}{\partial z \partial k} & \frac{\partial^2 \Pi}{\partial k^2} & \frac{\partial^2 \Pi}{\partial k \partial i} \\ \frac{\partial^2 \Pi}{\partial z \partial i} & \frac{\partial^2 \Pi}{\partial k \partial i} & \frac{\partial^2 \Pi}{\partial i^2} \end{bmatrix} = \begin{bmatrix} -0.396 & -0.037 & 1.121 \\ -0.037 & -0.038 & 0.105 \\ 1.121 & 0.105 & -0.739 \end{bmatrix}$$

Step 4: Construct the quadratic approximation.

(4-1) The payoff function, $\Pi(z, k, i)$ is approximated around the steady state, $(\bar{z}, \bar{k}, \bar{i})$ by the second-order Taylor approximation.

$$\Pi(z, k, i) \simeq \bar{\Pi} + (W - \bar{W})^T \bar{J} + \frac{1}{2}(W - \bar{W})^T \bar{H}(W - \bar{W}) \tag{9}$$

where

T := the superscript to indicate the transpose of a matrix.

(4-2) The above return function is rewritten by grouping the scalar terms, the linear terms and the quadratic terms.

$$\Pi(z, k, i) \simeq (\bar{\Pi} - \bar{W}^T \bar{J} + \frac{1}{2}\bar{W}^T \bar{H}\bar{W}) + W^T(\bar{J} - \bar{H}\bar{W}) + \frac{1}{2}W^T \bar{H} W \tag{10}$$

(4-3) Rewrite the above equation in a quadratic form.

$$\Pi(z, k, i) \simeq [1 \;\; W^T] \begin{bmatrix} \bar{\Pi} - \bar{W}^T \bar{J} + \frac{1}{2}\bar{W}^T \bar{H}\bar{W} & \left(\frac{1}{2}(\bar{J} - \bar{H}\bar{W})\right)^T \\ \frac{1}{2}(\bar{J} - \bar{H}\bar{W}) & \frac{1}{2}\bar{H} \end{bmatrix} \begin{bmatrix} 1 \\ W \end{bmatrix} \tag{11}$$

(4-4) Rewrite the above equation with matrix.

$$\Pi(z, k, i) \simeq \begin{bmatrix} 1 & W^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12}^T \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} 1 \\ W \end{bmatrix} \tag{12}$$

where

$$Q_{11} = \bar{\Pi} - \bar{W}^T \bar{J} + \frac{1}{2}\bar{W}^T \bar{H}\bar{W} = -0.127,$$

$$Q_{12} = \frac{1}{2}(\bar{J} - \bar{H}\bar{W}) = \begin{bmatrix} 0.519 \\ 0.109 \\ -0.484 \end{bmatrix},$$

$$Q_{12}^T = [0.519 \;\; 0.109 \;\; -0.484], \text{and}$$

$$Q_{22} - \frac{1}{2}\bar{H} = \begin{bmatrix} -0.198 & -0.018 & 0.560 \\ -0.018 & -0.019 & 0.052 \\ -0.560 & 0.052 & -0.369 \end{bmatrix}.$$

(4-5) Further rewrite the above equation in a more compact form:

$$\Pi(z,k,i)_{1\times 1} \simeq [1 \quad W^T]_{1\times 4} Q_{4\times 4} \begin{bmatrix} 1 \\ W \end{bmatrix}_{4\times 1} \tag{13}$$

where

$$Q_{4\times 4} = \begin{bmatrix} Q_{11} & Q_{12}^T \\ Q_{12} & Q_{22} \end{bmatrix} = \begin{bmatrix} -0.127 & 0.519 & 0.109 & -0.484 \\ 0.519 & -0.198 & -0.018 & 0.560 \\ 0.109 & -0.018 & -0.019 & 0.052 \\ -0.484 & 0.560 & 0.052 & -0.369 \end{bmatrix} \tag{14}$$

Step 5: Compute the optimal value function by repeated iterations until convergence occurs.

(5-1) The value function set up for the repeated iterations on the Bellman's equation is:

$$\begin{aligned} V_{n+1}(Z_t, k_t) &= \underset{i_t}{Max}\{\Pi(z_t, k_t, i_t) + \beta E[V_n(z_{t+1}, k_{t+1}|z_t]\} \\ &= \underset{i_t}{Max}\left\{[1 \quad Q_t^T]Q\begin{bmatrix} 1 \\ W_t \end{bmatrix} + \beta E[V_n(z_{t+1}, k_{t+1})|z_t\right\} \end{aligned} \tag{15}$$

subject to $k_{t+1} = (1-\delta)k_t + i_t$,
$z_{t+1} = \rho z_t + \epsilon_{t+1}$,
k_0 and z_0 given.

(5-2) The converged n-th iterated value function V_n has to be a quadratic concave function which is expressed in a matrix form as:

$$\begin{aligned} V_n(z_t, k_t) &= [1 \quad z_t \quad k_t]P_n \begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix} \\ &= F_t^T P_n F_t. \end{aligned}$$

The initial value $V_0(z_t, k_t)$ for the n-th iteration is given as:

$$V_0(z_t, k_t) = [1 \quad z_t \quad k_t] P_0 \begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix}$$
$$= F_t^T P_0 F_t \leq \ 0.$$

where

$F_t := [1 \quad z_t \quad k_t]^T$, the ordered column vector,

$P_0 :=$ any initial symmetric and negative semi-definite matrix, $P_0 = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$ as an example,

$P_n :=$ the n-th iterated symmetric matrix,

$V_0(\cdot) :=$ the initial value function for further iteration,

$V_n(\cdot) :=$ the n-th iterated value function.

Rewrite equation (15) in a quadratic form as:

$$V_n(z_t, k_t) = \underset{i_t}{Max} \left\{ [1 \quad W_t^T] Q \begin{bmatrix} 1 \\ W_t \end{bmatrix} + \beta E[F_{t+1}^T P_n F_{t+1} | z_t] \right\} \tag{16}$$

(5-3) In the case of this linear-quadratic approximation, the problem of solving the value function with the expectation operator in equation (16) is drastically simplified by applying the certainty equivalence principle.[6] This is a special property of the optimal linear regulator problem (Sargent, 1987). It is due to the quadratic nature of the payoff function and the linear nature of the constraints. The constraints in this problem are the equation of motion for capital and the first order Markov process of a series of technological shock. With the certainty equivalence principle, the value function is rewritten as:

$$V_{n+1}(z_t, k_t) = \underset{i_t}{Max} \left\{ [1 \quad W_t^T] Q \begin{bmatrix} 1 \\ W_t \end{bmatrix} + \beta F_{t+1}^T P_n F_{t+1} \right\} \tag{17}$$

subject to $k_{t+1} = (1-\delta) k_t + i_t,$

$z_{t+1} = \rho z_t,$

k_0 and z_0 given.

6) The optimal policy function that maximizes the value function is independent of the noise statistics of the problem. This feature is called the "certainty equivalence principle" (Sargent, 1987).

Step 6: Transform the value function into a quadratic form in a matrix $[1 \quad W^T]$.

(6-1) Rewrite the linear constraints such as the equation of motion for capital and the first order Markov process of technological shock in the following matrix form:

$$\begin{bmatrix} 1 \\ z_{t+1} \\ k_{t+1} \end{bmatrix}_{3\times1} = \begin{bmatrix} 1 \\ \rho z_t \\ (1-\delta)k_t + i_t \end{bmatrix}_{3\times1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 1-\rho & 1 \end{bmatrix}_{3\times4} \begin{bmatrix} 1 \\ z_t \\ k_t \\ i_t \end{bmatrix}_{4\times1}$$

$$\therefore \; F_{t+1\;3\times1} = B_{3\times4} \begin{bmatrix} 1 \\ W_t \end{bmatrix}_{4\times1}$$

(6-2) Substitute $F_{t+1} = B[1 \quad W_t^T]^T$ and $F_{t+1}^T = [1 \quad W_t^T]B^T$ into equation (17).

$$V_{n+1}(z_t, k_t) = \underset{i_t}{Max}\left\{[1 \quad W_t^T]Q\begin{bmatrix} 1 \\ W_t \end{bmatrix} + \beta[1 \quad W_t^T]B^T P_n B\begin{bmatrix} 1 \\ W_t \end{bmatrix}\right\} \tag{18}$$

(6-3) Transform equation (18) to show clearly that the value function is quadratic in two state variables z_t and k_t and a decision variable i_t.

$$V_{n+1}(z_t, k_t) = \underset{i_t}{Max}\left\{[1 \quad W_t^T][Q + \beta B^T P_n B]\begin{bmatrix} 1 \\ W_t \end{bmatrix}\right\} \tag{19}$$

$$= \underset{i_t}{Max}\left\{[1 \;\; z_t \;\; k_t \;\; i_t][Q + \beta B^T P_n B]\begin{bmatrix} 1 \\ z_t \\ k_t \\ i_t \end{bmatrix}\right\}$$

$$\therefore \; V_{n+1}(z_t, k_t) = \underset{i_t}{Max}\left\{[F_t^T \; i_t]_{1\times4}[Q_{4\times4} + \beta B_{4\times3}^T P_{n\;3\times3} B_{3\times4}]_{4\times4}\begin{bmatrix} F_t \\ i_t \end{bmatrix}_{4\times1}\right\}_{1\times1} \tag{20}$$

where

$[Q + \beta C^T P_n C]$:= a square symmetric matrix,
F := the ordered column vector, $F_t = [1 \;\; z_t \;\; k_t]^T$],
W := the vector of ordered state and decision variables, $W_t = [z_t \;\; k_t \;\; i_t]^T$.

Step 7: Derive the first order conditions for maximizing the value function.
(7-1) Partition the n-th iterated matrix M_n which is equal to $B^T P_n B$ for simplicity. The initial iterated matrix is also shown. M_n changes every iteration until convergence occurs.

$$M_0 = B^T P_0 B = \begin{bmatrix} \begin{pmatrix} -0.100 & 0.000 & 0.000 \\ 0.000 & -0.090 & -0.000 \\ 0.000 & -0.000 & -0.081 \end{pmatrix} & \begin{pmatrix} 0.000 \\ 0.000 \\ -0.090 \end{pmatrix} \\ \begin{pmatrix} 0.000 & 0.000 & -0.090 \end{pmatrix} & \begin{pmatrix} -0.100 \end{pmatrix} \end{bmatrix},$$

$$M_n = B^T P_n B = \begin{bmatrix} M_{FF\ 3\times3} & M^T_{Fi\ 3\times1} \\ M_{Fi\ 1\times3} & M_{ii\ 1\times1} \end{bmatrix}_{4\times4}.$$

(7-2) Partition matrix Q in equation (14). Note that Q remains invariant throughout the iterations.

$$Q = \begin{bmatrix} Q_{11} & Q^T_{12} \\ Q_{12} & Q_{22} \end{bmatrix}_{4\times4} = \begin{bmatrix} Q_{FF\ 3\times3} & Q^T_{Fi\ 3\times1} \\ Q_{Fi\ 1\times3} & Q_{ii\ 1\times1} \end{bmatrix}_{4\times4}$$

$$Q_{4\times4} = \begin{bmatrix} \begin{pmatrix} -0.127 & 0.519 & 0.109 \\ 0.519 & -0.198 & -0.018 \\ 0.109 & -0.018 & -0.019 \end{pmatrix} & \begin{pmatrix} -0.484 \\ 0.560 \\ 0.052 \end{pmatrix} \\ \begin{pmatrix} -0.484 & 0.560 & -0.052 \end{pmatrix} & \begin{pmatrix} -0.369 \end{pmatrix} \end{bmatrix} \tag{22}$$

where

$$Q_{11} = \bar{R} - \bar{W}^T \bar{T} + \frac{1}{2}\bar{W}^T \bar{H}\bar{T},$$

$$Q_{12} = \frac{1}{2}(\bar{J} - \bar{H}\bar{W}),$$

$$Q_{22} = \frac{1}{2}\bar{H}.$$

(7-3) Substitute the partitioned matrices Q and M_n into the value function in equation (20).

$$V_{n+1}(z_t, k_t) = \underset{i_t}{Max}\left\{[F_t^T \ i_t^T]\left\{\begin{bmatrix} Q_{FF} & Q_{Fi}^T \\ Q_{Fi} & Q_{ii} \end{bmatrix} + \beta\begin{bmatrix} M_{FF} & M_{Fi}^T \\ M_{Fi} & M_{ii} \end{bmatrix}\right\}\begin{bmatrix} F_t \\ i_t \end{bmatrix}\right\} \tag{23}$$

Further transform equation (23) in a quadratic form in $[F_t^T, i_t^T]$.

$$V_{n+1}(z_t, k_t) = \underset{i_t}{Max}\left\{[F_t^T \ i_t^T]\begin{bmatrix} Q_{FF} + \beta M_{FF} & Q_{Fi}^T + \beta M_{Fi}^T \\ Q_{Fi} + \beta M_{Fi} & Q_{ii} + \beta M_{ii} \end{bmatrix}\begin{bmatrix} F_t \\ i_t \end{bmatrix}\right\} \tag{24}$$

Further manipulate equation (24).

$$\begin{aligned} &V_{n+1}(z_t, k_t) = \\ &\underset{i_t}{Max}\left\{F_t^T[Q_{FF} + \beta M_{FF}]F + 2i_t^T[Q_{Fi} + \beta M_{Fi}] + i_t^T[Q_{ii} + \beta M_{ii}]i_t\right\} \end{aligned} \tag{25}$$

(7-4) Find the first order conditions by differentiating equation (25) with respect to i_t^T and setting the equation equal to zero. The concavity of the value function $V_n(.)$ guarantees that the first-order necessary conditions are also the sufficient conditions.

$$2[Q_{Fi} + \beta M_{Fi}]F + 2[Q_{ii} + \beta M_{ii}]i_t = 0 \tag{26}$$

Step 8: Compute the n-th iterated of linear decision rule, $i_n(z_t, k_t)$.
(8-1) The linear investment decision rule derived from equation (26) is:

$$\begin{aligned} i_n(z_t, k_t) &= -\left[(Q_{ii} + \beta M_{ii})^{-1}(Q_{Fi} + \beta M_{Fi})\right]_{1\times 3} F_{3\times 1} \\ &= J_{n\ 1\times 3}^T \begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix}_{3\times 1} \end{aligned} \tag{27}$$

where

J_n = the vector of the coefficients of the decision rules.

(8-2) The 707th iterated converged decision rule corresponding to the value function shown in appendix A is:

$$i_{707}(z_t k_t) = J^T_{707\ 1\times 3}\begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix}_{3\times 1} \tag{28}$$

$$= [0.498320125\ 0.860740175\ -0.04105213]\begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix}$$

(8-3) When z and k are at the steady state values $\bar{z}$=0, and $\bar{k} = 3.532878917156419$ as calculated in (2-4) in **Step 2**, the converged decision rule at the 707th iteration in equation (28) produces the steady-state optimal investment value which exactly matches the steady state value derived for the original nonlinear planner's problem in section (2-4) in **Step 2**. This check indicates that the converged investment decision rule is consistent. In other words, the steady-state point in the $z-k-i$ surface is on the surface of the optimal policy function shown in figure A-1 in appendix A.

$$i_{707}(z_t, k_t) = 0.353287891715641 = \bar{i}.$$

Step 9: Derive the value function.

(9-1) Substitute the investment decision rule derived in equation (27) into (25). The value function V_{n+1} becomes a quadratic form in $F_t = [1, z_t^T, i_t^T]$ which is expressed in a matrix form as:

$$V_{n+1}(z_t, k_t) = F_t^T[Q_{FF} + \beta M_{FF} - (Q_{Fi} + \beta M_{Fi})^T (Q_{ii} + \beta M_{ii})^{-1}(Q_{Fi} + \beta M_{Fi})]F_t \tag{29}$$

(9-2) The quadratic concave value function in equation (29) is restated in equation (30) where P_{n+1} is a symmetric matrix of order 3×3 as expressed in equation (31).[7)]

$$V_{n+1}(z_t, k_t) = [1\ z_t\ k_t]_{1\times 3} P_{n+1\ 3\times 3}\begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix}_{3\times 1} \tag{30}$$

7) At the beginning of the iteration, P_0 is a symmetric negative semi-definite matrix and the corresponding value function V_0 is nonpositive.

$$\because \quad P_{n+1} = Q_{FF} + \beta M_{FF} - (Q_{Fi} + \beta M_{Fi})^{T} (Q_{ii} + \beta M_{ii})^{-1} (Q_{Fi} + \beta M_{Fi}) \qquad (31)$$

(9-3) The 707th iterated converged function $P^* = P_{707}$ is :

$$P^{*}_{3\times3} = P_{707\ 3\times3} = \begin{bmatrix} -0.40246875 & 8.083920048 & 0.736916091 \\ 8.083920048 & 1.002874359 & -0.19152701 \\ 0.736916091 & -0.19152701 & -0.08186399 \end{bmatrix} \qquad (32)$$

The specified level of the n-th iterated relative error[8] in a matrix norm is

$$\frac{\| A_n - P_n \|}{\| A_n \|} = \frac{\sqrt{\sum_{j=0}^{2}\sum_{j=0}^{2} | A_{ij} - P_{ij} |^2}}{\sqrt{\sum_{i=0}^{2}\sum_{j=0}^{2} | A_{ij} |^2}} = 9.866e{-16} < Tolerance, \qquad (33)$$

$$\because \quad Tolerance = 1.0e - 15$$

where the initial matrix P_0 is assumed earlier and the initial auxiliary matrix $A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is assumed.

(9-4) Given the initial values for P_0 and A_0, the iterated function P_n turns out that $P_n = P_{n+1} = P^*$ at n = 707 for the desired tolerance level (the relative error) of 1.0e-15. The process of the convergence of matrix P_n is shown in appendix C. As P_n converges, the value function V_n is also converged as $V_n = V_{n+1} = V^*$. From equation (30), the estimated value for the optimal value function V^* is stated as a quadratic concave function the figure of which is shown in appendix B.

$$V_{707}(z_t, k_t) = [1\ z_t\ k_t] \begin{bmatrix} -0.40246875 & 8.083920048 & 0.736916091 \\ 8.083920048 & 1.002874359 & -0.19152701 \\ 0.736916091 & -0.19152701 & -0.08186399 \end{bmatrix} \begin{bmatrix} 1 \\ z_t \\ k_t \end{bmatrix} \qquad (34)$$

8) In Higham (p.5,1996) the relative error is desired for the accuracy of an approximation. The number of correct significant digits provides a useful way in which to think about the accuracy of approximation. However, Higham also stated that the relative error is a more precise measure and is base independent. Therefore an estimate or bound for the relative error should be provided whenever an approximate answer to a problem is found.

Step 10: Plot the time series of the optimal investment rule and some other related variables.

(10-1) Generate a series of independently and identically distributed random variable $\varepsilon_t \sim (0, \sigma^2) \simeq (0.000662, 0.003646)$.

(10-2) Give initial values to $k_0 = 0.1$ and $z_0 = 0.1$. Then iterate all the constraints shown in 5) in section 2.2 with the derived decision (investment) rule in equation (28) to produce the artificial time series (the equilibrium paths) of the relevant variables over 120 months (10 years) which clearly illustrates the economy's upturns and downturns as shown in appendix D. For example, when investment is high, capital stock, output, and consumption grow. When investment plummets, so do capital stock, output, and consumption.

4. CONCLUSION

The objective of this paper is to describe and validate an algorithm for solving a recursive linear- quadratic social planner's problem. This problem is a recurring one, particularly in growth and real business cycle literature. The computational details are provided in steps for the computational replication and accuracy of the reported simulation results: (1) numerical results are provided in intermediate steps as well as a final step, and (2) the relative error of the finite computational precision is also provided in a matrix norm. The computational example for this paper is drawn from Díaz-Giménez (1999), which provides the MATLAB software program written by J. Duran. I reprogrammed the same model in C-language with my own sub-routines and verified the reliability of the simulation results with greater precision (in double precision) than with the results obtained by J. Duran. I wholeheartedly agree with the view of McCullough and Vinod (1999) that computational aspects of running economic simulation models must be clear so that other researchers can replicate and verify the reported simulation results. Above all, computational accuracy should not be taken for granted.

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APPENDIX A

Figure A-1. The steady state point on the surface of the optimal policy function (optimal decision rule) $i^{*} = i_{707}(z_t, k_t)$ in equation (28) in the $z-k-i$ surface.

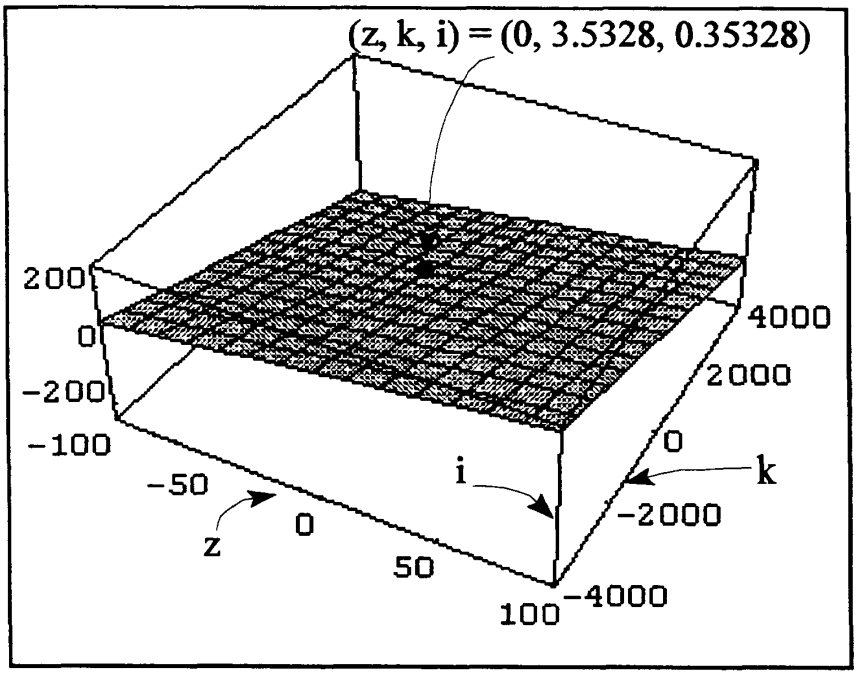
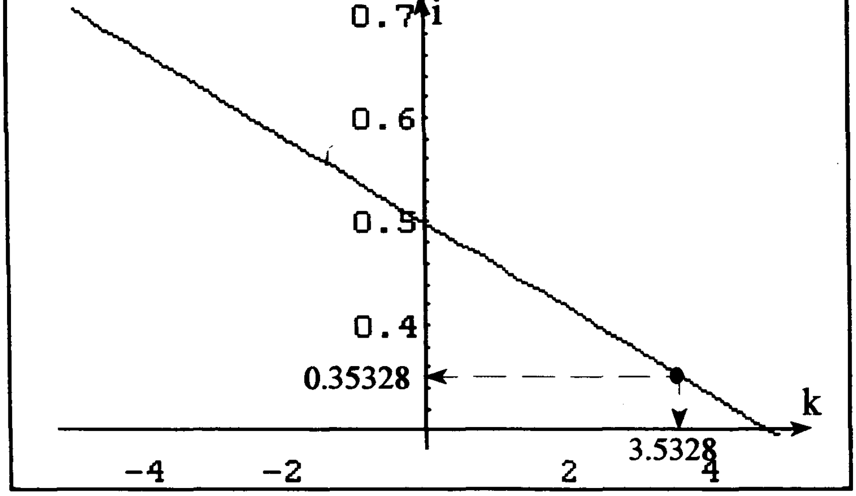


Figure A-2. When $z_t = 0$, the surface of the optimal policy function (optimal decision rule) in equation $i^{*} = i_{707}(z_t, k_t)$ (28) becomes the curve $i^{*} = i_{707}(0, k_t)$ shown in the $i-k$ plane on which there is the steady state point.



APPENDIX B

Figure B-1. The steady state point on the surface of the optimal value function $V^* = V_{707}(z_t, k_t)$ inequation (34) in the $z-k-V$ surface.

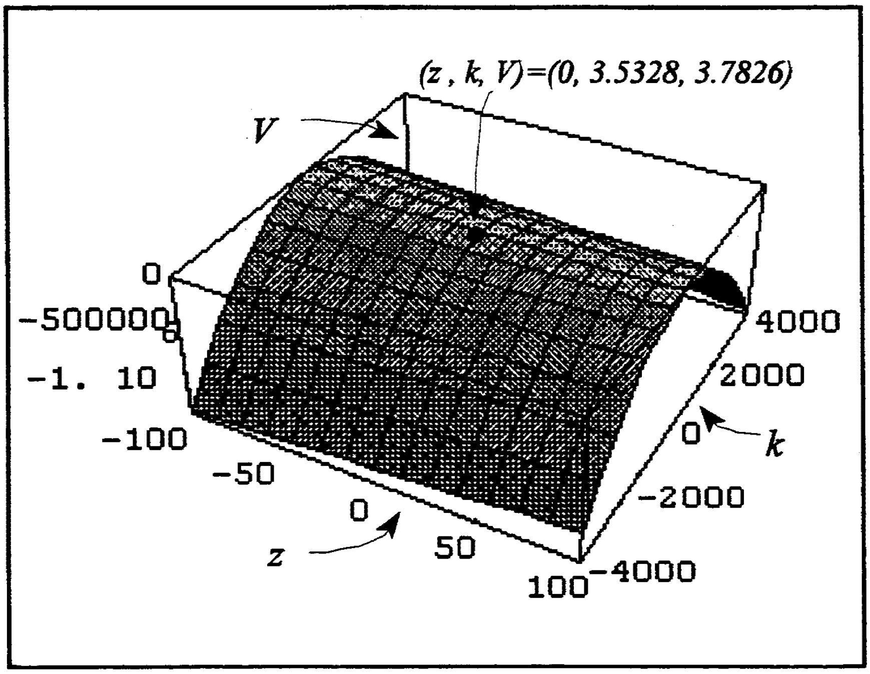
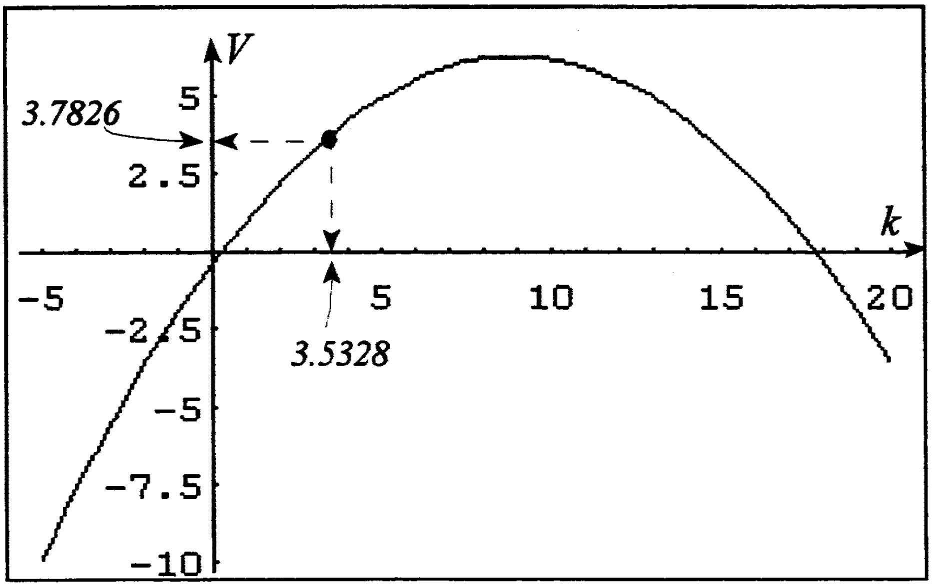


Figure B-2. When $z_t = 0$, the surface of the optimal value function $V^* = V_{707}(z_t, k_t)$ in equation (34) becomes the curve $V^* = V_{707}(0, k_t)$ shown in the $V-k$ plane on which there is steady state point.



APPENDIX C

The Process of the Convergence of Matrix P^* from P_0 to P_{706}

/* Note: The underlined numbers indicate the convergence of the numbers at the time of iteration t. */

t = 0 (*The given initial values*)

P[0][0] = -0.1
P[0][1] = 0
P[0][2] = 0
P[1][0] = 0
P[1][1] = -0.1
P[1][2] = 0
P[2][0] = 0
P[2][1] = 0
P[2][2] = -0.1

t = 50

P[0][0] = -0.52343286977230963
P[0][1] = 7.94679731432382219
P[0][2] = **0.736**87829582542119
P[1][0] = 7.94679731432382219
P[1][1] = **1.00**347273959647909
P[1][2] = **-0.19152**693933235002
P[2][0] = **0.736**87829582542119
P[2][1] = **-0.1915**2693933235002
P[2][2] = **-0.081863988**28688887

J[0][0] = **0.4982**2191305488894
J[1][0] = **0.86074**036391953823
J[2][0] = **-0.04105213**907909572

NormA_P/NormA = 1.705e-03 (*The relative error in a matrix norm*)

t = 100

P[0][0] = **-0.4**1821493956881145
P[0][1] = **8.08**254608964772814
P[0][2] = **0.7369160**8893399441
P[1][0] = **8.08**254608964772814
P[1][1] = **1.002874**81997499350
P[1][2] = **-0.191527012072**20799
P[2][0] = **0.7369160**8893399441
P[2][1] = **-0.191527012072**20804
P[2][2] = **-0.08186398791987863**

J[0][0] = **0.4983201**1863398001
J[1][0] = **0.86074017490**465848
J[2][0] = **-0.04105213812541784**

NormA_P/NormA = 5.917e-05

t = 200

P[0][0] = **-0.402**73439400798061
P[0][1] = **8.0839**1991029974122
P[0][2] = **0.73691609138507430**
P[1][0] = **8.0839**1991029974122
P[1][1] = **1.00287435879**902542
P[1][2] = **-0.19152701207255385**
P[2][0] = **0.73691609138507430**
P[2][1] = **-0.19152701207255390**
P[2][2] = **-0.08186398791987863**

J[0][0] = **0.49832012500312300**
J[1][0] = **0.86074017490375965**
J[2][0] = **-0.04105213812541784**

NormA_P/NormA = 9.596e-07

t = 500

P[0][0] = **-0.40246875**175261798
P[0][1] = **8.08392004753463134**
P[0][2] = **0.73691609138507430**
P[1][0] = **8.08392004753463134**
P[1][1] = **1.00287435879875320**
P[1][2] = **-0.19152701207255385**
P[2][0] = **0.73691609138507430**
P[2][1] = **-0.19152701207255390**
P[2][2] = **-0.08186398791987863**

J[0][0] = **0.49832012500312300**
J[1][0] = **0.86074017490375965**
J[2][0] = **-0.04105213812541784**

NormA_P/NormA = 4.608e-12

t = 700

P[0][0] = **-0.40246875047751413**
P[0][1] = **8.08392004753463134**
P[0][2] = **0.73691609138507430**
P[1][0] = **8.08392004753463134**
P[1][1] = **1.00287435879875320**
P[1][2] = **-0.19152701207255385**
P[2][0] = **0.73691609138507430**
P[2][1] = **-0.19152701207255390**
P[2][2] = **-0.08186398791987863**

J[0][0] = **0.49832012500312300**
J[1][0] = **0.86074017490375965**
J[2][0] = **-0.04105213812541784**

```
NormA_P/NormA = 1.309e-15

t = 707

P[0][0] = -0.40246875047742392
P[0][1] = 8.08392004753463134
P[0][2] = 0.73691609138507430
P[1][0] = 8.08392004753463134
P[1][1] = 1.00287435879875320
P[1][2] = -0.19152701207255385
P[2][0] = 0.73691609138507430
P[2][1] = -0.19152701207255390
P[2][2] = -0.08186398791987863

J[0][0] = 0.49832012500312300
J[1][0] = 0.86074017490375965
J[2][0] = -0.04105213812541784

NormA_P/NormA = 9.866e-16

/*******************The end of the output file******************************/
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APPENDIX D

Figure D-1. Exogenous state variable $z[t]$ over 120 months (10 years)

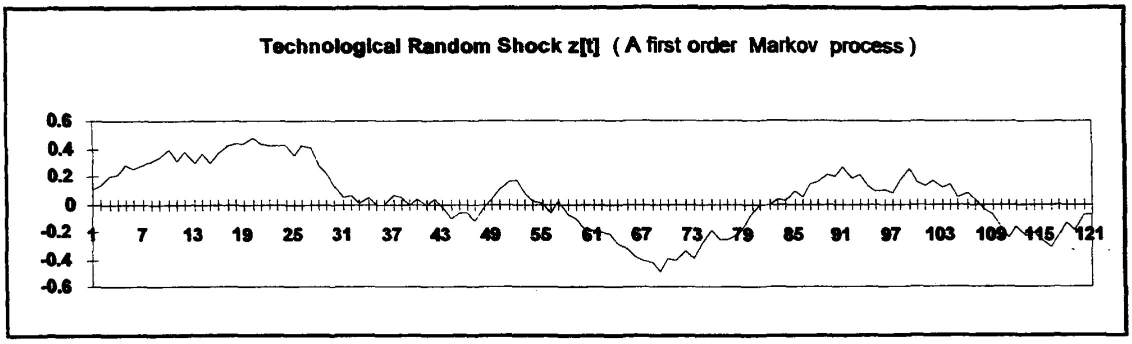

Figure D-2. Optimal investment schedule $i[t]$ over 120 months (10 years)

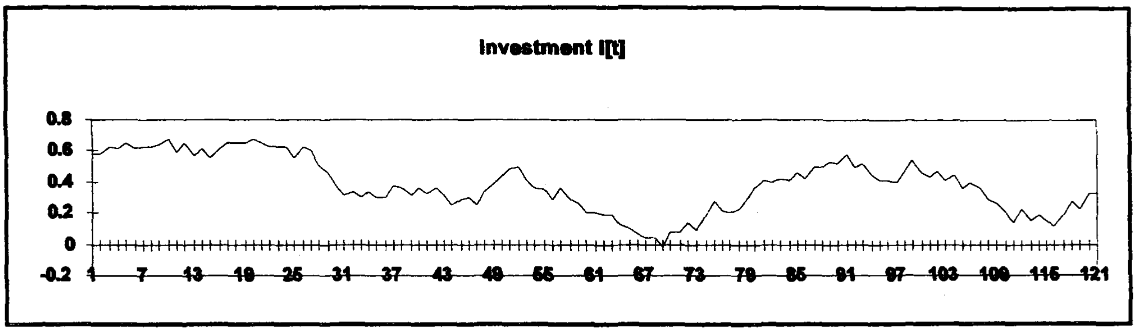

Figure D-3. Endogenous state variable $k[t]$ over 120 months (10 years)

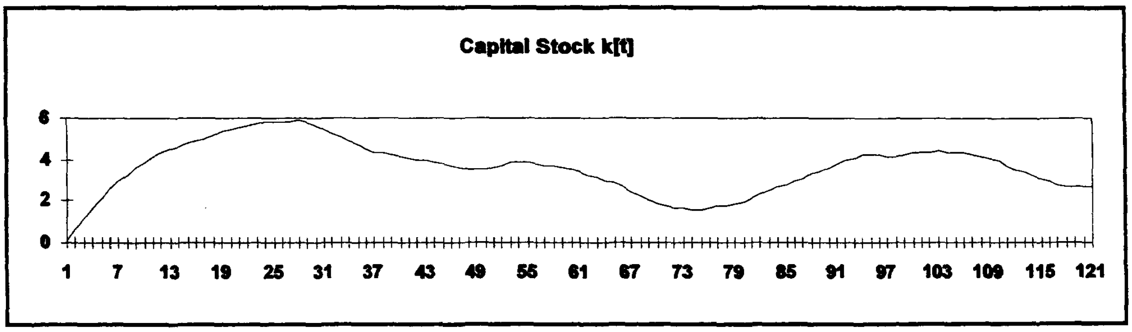

Figure D-4. Output $y[t]$ over 120 months (10 years)

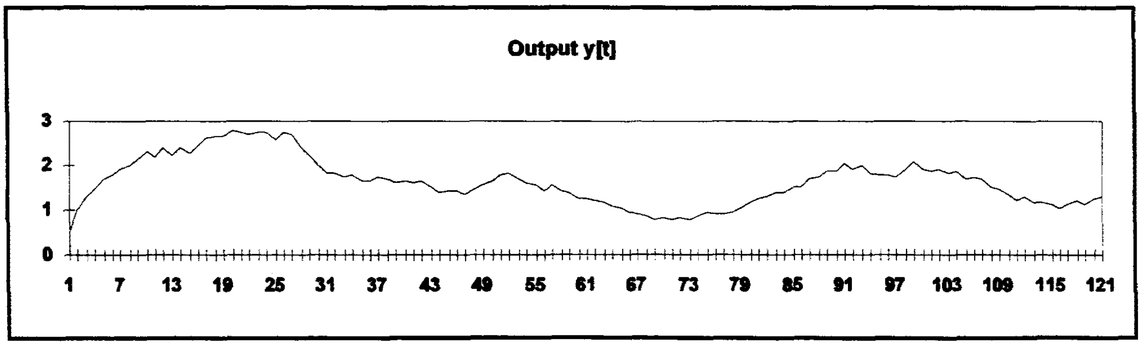

Figure D-5. Consumption $c[t]$ over 120 months (10 years)